

Inhomogenous model of crossing loops and multidegrees of some algebraic varieties

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We consider a quantum integrable inhomogeneous model based on the Brauer algebra $B(1)$ and discuss the properties of its ground state eigenvector. In particular we derive various sum rules, and show how some of its entries are related to multidegrees of algebraic varieties.

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1. Introduction

Recently, a new connection between quantum integrable models and combinatorics has emerged. This relation can be traced back to the idea, as expressed e.g. in [1], that in stochastic integrable processes, due to the existence of a simple ground state eigenvalue (without any finite-size corrections), the entries of the ground state are integers and must have some combinatorial significance. This idea was based on experience with a particularly succesful case: the model of non-crossing loops related to the Temperley–Lieb algebra $TL(1)$, whose special properties [2,3] led Razumov and Stroganov to conjecture the combinatorial significance of *each* entry of the ground state [4]. This conjecture has generated a lot of activity (see for example references in [5]) but has not been proved yet in its full generality.

The latest model that falls into the framework described above is the model of crossing loops proposed by de Gier and Nienhuis in [6], which is related to the Brauer algebra $B(1)$ and to standard integrable models with symmetry $OSp(p|2m)$ [7,8], $p - 2m = 1$. By abuse of language, as in the non-crossing case, we shall call this model the “ $O(1)$ ” crossing loop model. The novelty in the work [6] is that the entries of the ground state are integers that do not appear to be obviously related to statistical mechanics, but rather belong to the realm of enumerative geometry. Indeed some of them are conjectured to be degrees of algebraic varieties that appear in work of Knutson [9] revolving around the commuting variety. The present article tries to shed some light on the origin of these numbers in the model. In particular we shall see how the algebra of the model naturally leads to an action of the symmetric group as divided difference operators, which have well-known meaning in the context of Schubert calculus.

Our work is motivated by recent progress in understanding the model of non-crossing loops [5] for the similar Razumov–Stroganov conjecture. The idea of [5] is to make better use of the integrability of the model. It involves in particular the introduction of inhomogeneities (spectral parameters), which give a much more powerful tool to study the ground state, whose coefficients become polynomials in these variables. Here, we shall try to do the same to the $O(1)$ crossing loop model. As in [5], our results include multi-parameter sum rules for the entries of the ground state vector; we find in fact two different sum rules, one for the sum of all entries, and one for the sum in the so-called permutation sector, in which the entries clearly play a special role: these are precisely the coefficients which are conjecturally related to degrees of varieties. In fact we show that this connection is much

deeper and that the full polynomial entries are related to so-called multidegrees. We also prove some conjectured properties formulated in [6], involving factorizability of the ground state vector entries.

The paper is organized as follows. In Section 2 we introduce the model and its ground state eigenvector. Section 3 contains general factorization properties of the entries of the ground state, as well as their construction in terms of divided difference operators in the space of polynomials. Section 4 analyzes in detail the special case of so-called “permutation patterns”, which is also the focus of [6]; we formulate a conjecture that relates some of its entries to (multi)degrees of some algebraic varieties, prove some results including a sum rule, and give a sketch of proof of this generalized de Gier–Nienhuis conjecture. Section 5 concerns recursion relations and the sum rule for all entries. A few concluding remarks are gathered in section 6. The appendices contain some explicit data for $n = 2, 3, 4$.

2. The inhomogeneous $O(1)$ crossing loop model: transfer matrix and ground state vector

The $O(1)$ crossing loop model is based on the following solution to the Yang–Baxter equation, expressed as a linear combination of generators of the Brauer algebra $B_{2n}(1)$. These are the identity I , the “crossing” operators f_i , and the generators e_i of the Temperley–Lieb algebra $TL_n(1)$, $i = 1, 2, \dots, 2n$, with the pictorial representations

$$I = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \end{array}, \quad f_i = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}, \quad e_i = \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagdown \quad \diagup \end{array}$$

and acting vertically on the vector space generated by crossing link patterns, that is chord diagrams of $2n$ labeled points around a circle, connected by pairs via straight lines across the inner disk. We denote by CP_n the set of these (crossing) link patterns on $2n$ points, with cardinality $|CP_n| = (2n - 1)!!$. A simple way of indexing these link patterns is via permutations of \mathcal{S}_{2n} with only 2-cycles (fixed-point free involutions), each cycle being made of the labels of the two points connected via a chord. The pictorial representation above makes it straightforward to derive the $B_{2n}(1)$ Brauer algebra relations:

$$\begin{aligned} e_i^2 &= e_i, & f_i^2 &= I, & e_i e_{i\pm 1} e_i &= e_i, & f_i f_{i+1} f_i &= f_{i+1} f_i f_{i+1}, \\ [e_i, e_j] &= [e_i, f_j] = [f_i, f_j] = 0 \text{ if } |i - j| > 1, & f_i e_i &= e_i f_i = e_i \end{aligned} \tag{2.1}$$

Looking for a solution for a face transfer matrix operator $X_i(u) = a(u)I + b(u)f_i + c(u)e_i$ to the Yang–Baxter equation

$$X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u) \quad (2.2)$$




further fixed by the normalization $X_i(0) = I$, we find the solution

$$X_i(u) = (1-u)I + \frac{u}{2}(1-u)f_i + ue_i, \quad (2.3)$$

unique up to scaling of u , as a direct consequence of the relations (2.1). The solution (2.3) also satisfies the unitarity relation

$$X_i(u)X_i(-u) = (1-u^2)(1-u^2/4)I \quad (2.4)$$

This solution appeared first in [10], and was further studied in [7], and shown to be related to vertex models based on orthosymplectic groups.

We now introduce an inhomogeneous integrable model based on the above solution of the Yang–Baxter equation. It is defined on an infinite cylinder of square lattice of perimeter $2n$ represented as an infinite strip of width $2n$ glued along its two borders. A configuration of the model is defined by assigning the plaquettes , , or  to each elementary face of the cylinder, with certain weights.

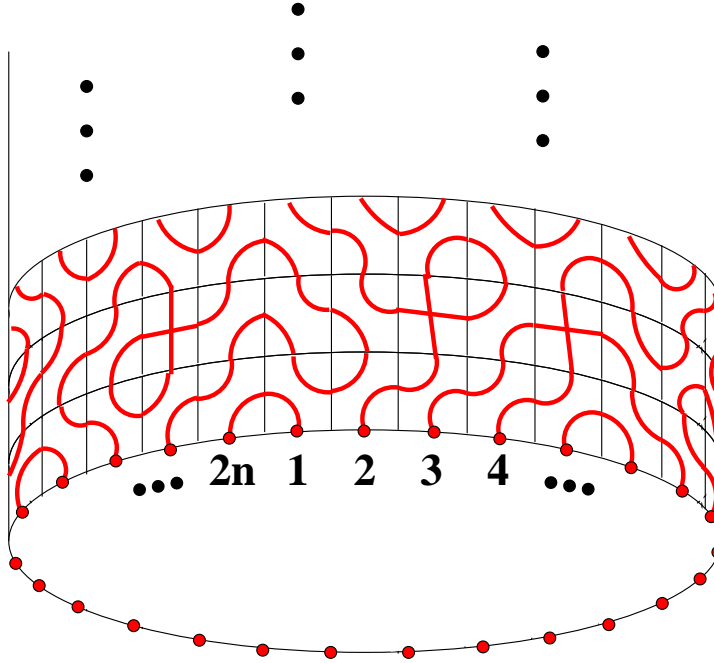


Fig. 1: A typical configuration of the crossing loop model on a semi-infinite cylinder of square lattice with perimeter $2n$.

In the transfer matrix approach, one considers a semi-infinite cylinder (see Fig. 1). The space of states then represents the pattern of pair connectivity of the $2n$ labeled midpoints of the boundary edges of the semi-infinite cylinder via plaquette configurations of the model. Finally, the transfer matrix represents the addition of one row of plaquettes to the semi-infinite cylinder:

$$T_n(t|z_1, \dots, z_{2n}) = \prod_{i=1}^{2n} \left((1-t+z_i) \begin{array}{|c|} \hline \text{red arcs} \\ \hline \end{array} + \frac{(t-z_i)(1-t+z_i)}{2} \begin{array}{|c|} \hline \text{red cross} \\ \hline \end{array} + (t-z_i) \begin{array}{|c|} \hline \text{red arcs} \\ \hline \end{array} \right) \quad (2.5)$$

where the weights depend on the label i of the site, and correspond to a tilted version of the operators X_i of Eq. (2.3). The parameter t , which is independent of the row, plays no role in what follows due to the commutativity property

$$[T_n(t), T_n(t')] = 0 \quad (2.6)$$

itself a direct consequence of the Yang–Baxter equation.

For values of z_i and t such that $0 < t - z_i < 1$, the weights are strictly positive and can be interpreted as unnormalized probabilities, and the transfer matrix as an unnormalized matrix of transition probabilities. Conservation of probability can be expressed in the following way: define the linear form v_n with entries in the canonical basis $v_\pi = 1$ for all $\pi \in CP_n$. Then summing the weights in Eq. (2.5), we obtain

$$v_n T_n(t|z_1, \dots, z_{2n}) = v_n \prod_{i=1}^{2n} \left(1 - \frac{1}{2}(t - z_i) \right) (1 + t - z_i) \quad (2.7)$$

This means that $\prod_{i=1}^{2n} (1 - \frac{1}{2}(t - z_i))(1 + t - z_i)$ is an eigenvalue of T_n (with left eigenvector v_n), and there must exist a right eigenvector:

$$\left(T_n(t|z_1, \dots, z_{2n}) - \prod_{i=1}^{2n} \left(1 - \frac{1}{2}(t - z_i) \right) (1 + t - z_i) I \right) \Psi_n(z_1, \dots, z_{2n}) = 0 \quad (2.8)$$

In the aforementioned range, Eqs. (2.7) and (2.8) are nothing but Perron–Frobenius eigenvector equations for the transpose of T_n and for T_n , and the entries Ψ_π of Ψ_n are interpreted, up to normalization, as the equilibrium probabilities, in random configurations of the model on a semi-infinite cylinder, that the boundary vertices be connected according to π .

As T_n is polynomial, we may assume that Ψ_n is also a polynomial of the z_i (whose entries are non-identically-zero due to the Perron–Frobenius property). Since we can always

factor out the GCD of the entries Ψ_π , we assume that they are coprime. The main purpose of the present article is the investigation of these entries. A special case, extensively studied in [6], corresponds to choosing the z_i to be all equal. In this “homogeneous” case, $T_n(t)$ commutes with the Hamiltonian $H_n = \sum_{i=1}^{2n} (3 - 2e_i - f_i)$, and Ψ_n is the null eigenvector of H_n . It was conjectured in [6] that with proper normalization, the entries of Ψ_n may be chosen to be all non-negative integers, the smallest of which is 1. Here we use the latter condition to fix the remaining arbitrary numerical factor in the normalization of the entries, so that it coincides in the homogeneous case with that of [6].

Before going into specifics, let us mention a preliminary property satisfied by the entries of Ψ_n . Our semi-infinite cylinder problem is clearly invariant under rotation by one lattice step. Denoting by $\rho = f_{2n-1}f_{2n-2}\dots f_1$ the corresponding rotation operator acting on the crossing link patterns by cyclically shifting the labels $i \rightarrow i+1$, we have the relation $T_n(t|z_2, \dots, z_{2n}, z_1)\rho = \rho T_n(t|z_1, \dots, z_{2n})$, from which we deduce that $\rho\Psi_n(z_1, \dots, z_{2n}) = \lambda\Psi_n(z_2, \dots, z_{2n}, z_1)$. Noting that Ψ_n is generically non-zero due to the Perron–Frobenius property, and that λ takes discrete values $\lambda^{2n} = 1$ and must therefore be independent of the z_i , we immediately get that $\lambda = 1$ in the range where $\Psi_\pi > 0$, henceforth the entries of Ψ_n satisfy the following cyclic covariance relation:

$$\Psi_{\rho \cdot \pi}(z_2, z_3, \dots, z_{2n}, z_1) = \Psi_\pi(z_1, z_2, \dots, z_{2n}) \quad (2.9)$$

Similarly, one can prove a reflection relation: if r exchanges i and $2n+1-i$,

$$\Psi_{r \cdot \pi}(-z_{2n}, -z_{2n-1}, \dots, -z_1) = \Psi_\pi(z_1, z_2, \dots, z_{2n}) \quad (2.10)$$

3. Factorization and degree

We now establish factorization properties of the transfer matrix T_n and of its eigenvector Ψ_n . Note that this section (as well as Sect. 5 below) possesses some strong similarities with Sect. 3 of [5], though the model under consideration is different. It is sometimes convenient to use the following pictorial representations for the matrix $X_i(t-z)$ and for the transfer matrix:

$$X_i(u) = \begin{array}{c} \diagup \\ \mathbf{u} \\ \diagdown \end{array}, \quad T_n(t|z_1, \dots, z_{2n}) = \begin{array}{c} \text{---} t-z_1 \text{---} t-z_{2n} \text{---} \\ \text{---} t-z_2 \text{---} \\ \vdots \end{array} \quad (3.1)$$

In this language, the Yang–Baxter and unitarity relations read respectively:

$$\text{Diagram 1} = \text{Diagram 2} \quad \text{and} \quad \text{Diagram 3} = (1 - u^2)\left(1 - \frac{u^2}{4}\right) \quad (3.2)$$

In all that follows, due to periodic boundary conditions indices are meant modulo $2n$ ($2n + 1 \equiv 1$).

3.1. Vanishings and factorizations

Let us show a first intertwining property:

Lemma 1. *The matrices $T_n(t|z_1, \dots, z_i, z_{i+1}, \dots, z_{2n})$ and $T_n(t|z_1, \dots, z_{i+1}, z_i, \dots, z_{2n})$ are intertwined by $X_i(z_{i+1} - z_i)$, namely*

$$\begin{aligned} T_n(t|z_1, \dots, z_i, z_{i+1}, \dots, z_{2n}) X_i(z_{i+1} - z_i) \\ = X_i(z_{i+1} - z_i) T_n(t|z_1, \dots, z_{i+1}, z_i, \dots, z_{2n}) \end{aligned} \quad (3.3)$$

Proof. This is a direct consequence of the Yang–Baxter relation and reads pictorially:

$$\dots \text{Diagram 1} \dots = \dots \text{Diagram 2} \dots \quad (3.4)$$

□

We now remark that at the value 1 of the parameter, the face transfer matrix reduces to $X_i(1) = e_i$. This means that for $z_{i+1} = z_i + 1$, the above transfer matrices say T and \tilde{T} satisfy $Te_i = e_i\tilde{T}$. When acting on $\tilde{\Psi}_n \equiv \Psi_n(\dots z_{i+1}, z_i, \dots)$ at $z_{i+1} = z_i + 1$, we get: $Te_i\tilde{\Psi}_n = \Lambda e_i\tilde{\Psi}_n$, with $\Lambda = \prod_{i=1}^{2n} (1 - \frac{1}{2}(t - z_i))(1 - z_i + t)$. Hence $e_i\tilde{\Psi}_n$ is a non-vanishing vector proportional to Ψ_n , and there exists a rational function α , such that $\Psi_n = \alpha e_i\tilde{\Psi}_n$. When written in components, this implies that whenever i and $i + 1$ are not connected via a “little arch” in a link pattern $\pi \in CP_n$, the entry Ψ_π vanishes when $z_{i+1} = z_i + 1$. We may extend this remark into a:

Proposition 1. *If the link pattern $\pi \in CP_n$ has no arch connecting a pair of points between labels i and j , then the entry Ψ_π vanishes for $z_j = z_i + 1$.*

The proof is already done in the case $j = i + 1$. For more distant points, we use a generalized intertwining property $TP = P\tilde{T}$, where P is a suitable product of X matrices. Using again the fact that $X(1) = e_i$, we see that at $z_j = z_i + 1$ the product of X forming P contains a factor e_i at the intersection between the lines i and j . We deduce that $\Psi_n = \alpha P\tilde{\Psi}_n$ has no non-vanishing entry with at least an arch linking two points between i and j . Indeed, by expanding the product of X_i that form P as a sum of products of f and e , we see that there is always at least one e_k in factor, for $i \leq k < j$, which results in the existence of an arch connecting two points inbetween i and j . \square

This shows that Ψ_π is divisible by $\prod_{i \leq k < l \leq j} (1 + z_k - z_l)$ (with obvious cyclic notations) for all pairs of points i and j satisfying the hypothesis of Prop. 1.

As a first application, let us consider the link pattern π_0 without any little arches, and the maximum number of crossings: $\pi_0(i) = i + n$, $i = 1, \dots, n$. For each variable z_i , we have a factor of $\prod_{j=i+1}^{i+n-1} (1 + z_i - z_j) \prod_{k=i+n+1}^{i-1} (1 + z_k - z_i)$. In total, this gives

$$\Psi_{\pi_0} = \Omega \prod_{\substack{1 \leq i < j \leq 2n \\ j-i < n}} (1 + z_i - z_j) \prod_{\substack{1 \leq i < j \leq 2n \\ j-i > n}} (1 + z_j - z_i) \quad (3.5)$$

where Ω is a polynomial yet to be determined. Apart from this, we find a polynomial of total degree $n(2n-1) - n = 2n(n-1)$ and partial degree $2n-2$ in each variable. We shall prove in the following that $\Omega = 1$.

3.2. Permutation of variables in Ψ_n and degree

Let us introduce a normalized version of X_i , which we denote by \check{R}_i :

$$\check{R}_i(z, w) = \frac{(1 - w + z)I + \frac{1}{2}(w - z)(1 - w + z)f_i + (w - z)e_i}{(1 - \frac{1}{2}(w - z))(1 + w - z)} \quad (3.6)$$

This matrix satisfies the usual unitarity relation $\check{R}_i(z, w)\check{R}_i(w, z) = I$.

Theorem 1. *The transposition of any two consecutive spectral parameters in Ψ_n is generated by the action of \check{R} :*

$$\Psi_n(\dots, z_i, z_{i+1}, \dots) = \check{R}_i(z_i, z_{i+1})\Psi_n(\dots, z_{i+1}, z_i, \dots) \quad (3.7)$$

Proof. To show this, we apply Lemma 1 to the vector $\Psi_n(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n})$. We find that $\Psi_n(z_1, \dots, z_{2n}) = \alpha_{n,i}(z_1, \dots, z_{2n})X_i(z_{i+1} - z_i)\Psi_n(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n})$, for some rational function $\alpha_{n,i}$. By the coprimarity assumption for the entries of Ψ_n , we deduce that $\alpha_{n,i}$ may have no zero, hence it reads $\alpha_{n,i} = 1/\beta_{n,i}$, for some polynomial $\beta_{n,i}$.

Moreover, iterating the above once more, we find that

$$\Psi_n = \check{R}_i(z_{i+1}, z_i)\check{R}_i(z_i, z_{i+1})\Psi_n = \frac{\beta_{n,i}(z_1, \dots, z_{2n})\beta_{n,i}(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n})}{(1 - \frac{1}{4}(z_i - z_{i+1})^2)(1 - (z_i - z_{i+1})^2)}\Psi_n \quad (3.8)$$

The only polynomials that satisfy this relation are

$$\beta_{n,i}(z_1, \dots, z_{2n}) = \left(1 + \epsilon_i(z_{i+1} - z_i)\right)\left(1 + \epsilon'_i \frac{1}{2}(z_i - z_{i+1})\right)\epsilon''_i \quad (3.9)$$

for $\epsilon_i, \epsilon'_i, \epsilon''_i = \pm 1$. These signs are further all fixed to be +1 by (i) expressing (3.9) when all $z_j = 0$ ($\epsilon''_i = 1$), (ii) expressing it when all $z_j \rightarrow \infty$ ($\epsilon_i \epsilon'_i = 1$), and (iii) by applying the Lemma 1 ($\epsilon_i = 1$). This yields Eq. (3.7). \square

More explicitly, Eq. (3.7) reads in components:

$$\begin{aligned} (1 + \frac{1}{2}(z_i - z_{i+1}))(1 - z_i + z_{i+1})\Psi_\pi(z_1, \dots, z_{2n}) \\ = (1 + z_i - z_{i+1})\Psi_\pi(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n}) \\ + \frac{1}{2}(z_{i+1} - z_i)(1 + z_i - z_{i+1})\Psi_{f_i \cdot \pi}(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n}) \\ + (z_{i+1} - z_i) \sum_{\pi', e_i \cdot \pi' = \pi} \Psi_{\pi'}(z_1, \dots, z_{i+1}, z_i, \dots, z_{2n}) \end{aligned} \quad (3.10)$$

This is a very efficient recursion relation, allowing to express all entries of Ψ_n in terms of the maximally crossing pattern entry. Indeed, two situations may occur for π :

(i) π has no little arch joining $(i, i+1)$. Then Eq. (3.10) translates into

$$\Psi_{f_i \cdot \pi}(z_1, \dots, z_{2n}) = \Theta_i \Psi_\pi(z_1, \dots, z_{2n}) \quad (3.11)$$

where the linear operator Θ_i acts on functions $F(z_1, \dots, z_{2n})$ as

$$\begin{aligned} \Theta_i F(z_1, \dots, z_{2n}) = 2 \frac{(1 + z_i - z_{i+1})F(z_{i+1}, z_i) - (1 - z_i + z_{i+1})F(z_i, z_{i+1})}{(z_i - z_{i+1})(1 - z_i + z_{i+1})} \\ - \frac{1 + z_i - z_{i+1}}{1 - z_i + z_{i+1}} F(z_{i+1}, z_i) \end{aligned} \quad (3.12)$$

where for simplicity we have only represented the arguments i and $i + 1$ of F (recall that periodic boundary conditions for indices are implied: $2n + 1 \equiv 1$). Note here that $\Theta_i \circ \Theta_i = I$, in agreement with $f_i^2 = I$, a simple consequence of the “gauge formula”

$$\Theta_i = (1 + z_i - z_{i+1}) (2\partial_i - \tau_i) \frac{1}{1 + z_i - z_{i+1}} \quad (3.13)$$

where τ_i and ∂_i are respectively the transposition and divided difference operators,¹ acting as

$$\begin{aligned} \tau_i F(z_i, z_{i+1}) &= F(z_{i+1}, z_i) \\ \partial_i F(z_i, z_{i+1}) &= \frac{F(z_{i+1}, z_i) - F(z_i, z_{i+1})}{z_i - z_{i+1}} \end{aligned} \quad (3.14)$$

with the obvious relations

$$\tau_i^2 = 1, \quad \partial_i^2 = 0, \quad \partial_i \tau_i = -\tau_i \partial_i \quad (3.15)$$

(ii) π has a little arch joining $(i, i + 1)$. Then Eq. (3.10) translates into

$$\sum_{\substack{\pi' \in CP_n \\ \pi' \neq \pi, \quad e_i \pi' = \pi}} \Psi_{\pi'}(z_1, \dots, z_{2n}) = \Delta_i \Psi_{\pi}(z_1, \dots, z_{2n}) \quad (3.16)$$

where the linear operator Δ_i acts as

$$\Delta_i F(z_1, \dots, z_{2n}) = (1 + z_i - z_{i+1}) \left(1 + \frac{1}{2}(z_{i+1} - z_i)\right) \frac{F(z_{i+1}, z_i) - F(z_i, z_{i+1})}{z_i - z_{i+1}} \quad (3.17)$$

Note also that $\Delta_i \circ \Delta_i = -\Delta_i$, in agreement with $(e_i - I)^2 = -(e_i - I)$.

Some remarks are in order. From the explicit form of Δ_i (3.17), one may think that its action on a polynomial increases the degree by one. However this is not the case if the largest total degree piece of the polynomial is symmetric under $z_i \leftrightarrow z_{i+1}$. Such a property will be found below (Lemma 2). Also, it is clear that the set of relations (3.11) and (3.16) is overdetermined. The compatibility between these equations is granted by the Yang–Baxter equation, that translates into relations between the Θ_i and Δ_i . Finally, we note that the system of equations (3.11) and (3.16) is equivalent to the eigenvector condition (2.8). To see that this system is sufficient to ensure (2.8), we note that in analogy with the Temperley–Lieb case of [5], we may construct yet another transfer matrix T'_n that

¹ Note the unusual sign convention for ∂_i , which will be fixed in Sect. 4 by renumbering variables in the opposite order.

commutes with T_n , and made of a particular product of matrices of the form $\check{R}_i(z_k, z_\ell)$. More precisely, we have $T'_n = \prod_{i=1}^n \prod_{j=1}^n \check{R}_{i+2j-2}(z_{2j-1}, z_{2i+2j-2})$, with products taken in the order indicated. As a direct consequence of the Yang–Baxter equation (3.4), we have the commutation relation $[T_n, T'_n] = 0$, hence if Ψ_n is eigenvector of T'_n with eigenvalue 1 (that is Perron–Frobenius eigenvector in appropriate ranges of the z_i), then it is also that of T_n . The set of conditions (3.11) and (3.16) precisely guarantee that Ψ_n is such an eigenvector of T'_n , hence they are sufficient to ensure (2.8).

As it turns out, we may generate all the entries Ψ_π for $\pi \in CP_n$ by acting with a number of Θ_i on Ψ_{π_0} . This is best seen by recalling that the link patterns $\pi \in CP_n$ can be considered as permutations of \mathcal{S}_{2n} with only 2-cycles. As such, f_i acts on π as conjugation of π by the elementary transposition $i \leftrightarrow i+1$, and generates the action of the whole symmetric group \mathcal{S}_{2n} . The well-known property that two permutations are conjugate if (and only if) they have the same cycle lengths implies that any π can be obtained from π_0 as $\pi = f_{i_1} \cdots f_{i_k} \cdot \pi_0$. We assume that $f_{i_{l+1}} \cdots f_{i_k} \cdot \pi_0$ does not have a little arch (i_l, i_l+1) for $1 \leq l \leq k$, i.e. exclude in such a decomposition any f_i that would act on a pattern with a little arch $(i, i+1)$ since such an action is trivial. We can therefore apply (i) above repeatedly, and express the corresponding entry of Ψ_n :

$$\Psi_\pi = \Theta_{i_1} \cdots \Theta_{i_k} \Psi_{\pi_0} \quad (3.18)$$

The procedure is illustrated in appendix B in the case $n = 3$.

The property (3.18) has an important consequence: by constructing explicitly the entries of Ψ , we fix their degree and prove that $\Omega = 1$ in Eq. (3.5):

Theorem 2. *One has:*

$$\Psi_{\pi_0} = \prod_{\substack{1 \leq i < j \leq 2n \\ j-i < n}} (1 + z_i - z_j) \prod_{\substack{1 \leq i < j \leq 2n \\ j-i > n}} (1 + z_j - z_i) \quad (3.19)$$

and all the entries of Ψ_n are polynomials of total degree $2n(n-1)$, and partial degree $2(n-1)$ in each z_i .

The proof goes as follows. Starting with the “minimal” candidate (3.19) for Ψ_{π_0} subject to the vanishing conditions of proposition 1 above, we must first show that we can construct all other entries Ψ_π via Eqs. (3.18) *independently of the choice of words* $f_{i_1} \cdots f_{i_k}$, and then check that these moreover satisfy Eqs. (3.16), and are coprime.

We first use Eq. (3.18) to express all the Ψ_π in terms of the minimal choice (3.19) for Ψ_{π_0} . It is easy to check that the factorization properties of proposition 1 are satisfied by all the Ψ_π thus constructed. As a consequence, all denominators in Eq. (3.12) are cancelled (see the reformulation (3.13)), and the action of Θ_i preserves the polynomial character, and the degree. The fact that the values of Ψ_π are independent of the choice of decomposition $\pi = f_{i_1} \cdots f_{i_k} \cdot \pi_0$ which appears in (3.18) is a consequence of the following properties:

- (i) The affine symmetric group relations satisfied by the Θ 's, namely $\Theta_i^2 = I$, $\Theta_i \Theta_{i+1} \Theta_i = \Theta_{i+1} \Theta_i \Theta_{i+1}$, and $\Theta_i \Theta_j = \Theta_j \Theta_i$ for $|i - j| > 1$, for all $i = 1, 2, \dots, 2n$ with cyclic boundary conditions, ensure that two words corresponding to the same permutation give rise to the same Ψ_π .
- (ii) As the Θ_i correspond only to the action of the f_i on permutations σ with n cycles of length 2 (the class $[2^n]$ of S_{2n}), while forbidding the action of f_i on permutations σ such that $\sigma(i) = i + 1$, we must also enforce the conditions that

$$\Theta_i \Theta_{n+i} \Psi_{\pi_0} = \Psi_{\pi_0} \quad (3.20)$$

for $i = 1, 2, \dots, n$.

The corresponding elements $f_i f_{i+n}$ of S_{2n} are indeed nothing but the remaining generators of the stabilizer subgroup of the class $[2^n]$, once the actions of f_i on permutations σ such that $\sigma(i) = i + 1$, $i = 1, 2, \dots, 2n$, have been forbidden.² Note that the latter condition has the net effect of wiping out any affine symmetric group relation that “winds” around the circle, such as for instance $f_2 f_3 \dots f_{2n} = I$, as its action on link patterns would involve at least one of the forbidden moves. The relations (3.20) are readily checked by using the expression (3.19). This defines the vector Ψ_n unambiguously.

To ensure that Ψ_n satisfies the eigenvector equation (2.8), we must still check that it satisfies the properties (3.16), which have not been used so far. This is done by induction on the number of crossings in π . Assume a link pattern π has a little arch in position $(i, i + 1)$, and that it satisfies Eq. (3.16). As $[\Delta_i, \Theta_j] = 0$ for $j \neq i - 1, i, i + 1$, we may act upon Eq. (3.16) with any such Θ_j , also such that π has no little arch at positions $(j, j + 1)$. This results in a new equation, in which $\pi, \pi' \rightarrow f_i \cdot \pi, f_i \cdot \pi'$. This allows to increase

² This action is exactly the “flat first Reidemeister move”. In other words to obtain a true representation of the Θ_i without restrictions one should consider link patterns decorated with extra “tadpoles”.

the number of crossings of π , hence the induction. We are left with the task of proving Eq. (3.16) for π with maximal number of crossings and with a little arch $(i, i+1)$. We first note that Ψ_{π_0} as given by (3.19) is cyclically invariant (under $z_i \rightarrow z_{i+1}$ cyclically), henceforth we need only check this property for $i = 2n$, say. We have to check that

$$\Delta_{2n} \Psi = \sum_{j=2}^n \left(\Psi + \Psi \right) \quad (3.21)$$

or equivalently

$$\Delta_{2n} \Theta_1 \Theta_2 \dots \Theta_{n-1} \Psi_{\pi_0} = (1 + \Theta_{2n}) \sum_{j=2}^n \Theta_1 \Theta_2 \dots \Theta_{j-2} \Theta_j \Theta_{j+1} \dots \Theta_{n-1} \Psi_{\pi_0} \quad (3.22)$$

where we have used the commutation of the Θ_i for distant enough indices, and the fact that $\Theta_i \Psi_{\pi_0} = \Theta_{n+i} \Psi_{\pi_0}$ for all i , a direct consequence of $f_i \cdot \pi_0 = f_{i+n} \cdot \pi_0$ for all i . Noting also that $1 + \Theta_i = \Delta_i \times 2/(1 + z_i - z_{i+1})$, and that Δ_i is proportional to ∂_i , we finally arrive at the condition that the quantity

$$\Phi_n = \left(\Theta_1 \Theta_2 \dots \Theta_{n-1} - \frac{2}{1 + z_{2n} - z_1} \sum_{j=2}^n \Theta_1 \Theta_2 \dots \Theta_{j-2} \Theta_j \Theta_{j+1} \dots \Theta_{n-1} \right) \Psi_{\pi_0} \quad (3.23)$$

must be annihilated by Δ_{2n} , hence by ∂_{2n} , or equivalently that Φ_n must be symmetric under $z_{2n} \leftrightarrow z_1$. We have found an explicit expression for Φ_n , displaying this invariance manifestly, namely

$$\Phi_n = \Psi_{\pi_0}^{(n-1)}(z_2, z_3, \dots, z_{2n-1}) \prod_{j=2}^n a_{1,j} a_{2n,j} a_{j+n-1,1} a_{j+n-1,2n} \quad (3.24)$$

where the first term is nothing but the expression for Ψ_{π_0} (3.19) but for the size $2n-2$, and with arguments counted cyclically from 2 to $2n-1$. We have also used the shorthand notation $a_{i,j} = 1 + z_i - z_j$. To prove (3.24), we first note that (3.22) may be equivalently rewritten as

$$\Phi_n = \left(\prod_{i=1}^{n-1} \Theta_i - \frac{2}{1 + z_{2n} - z_1} \right) \Psi_{\pi_0} \quad (3.25)$$

This is easily shown by induction, by noticing that Θ_i commutes with $2/a_{2n,j}$ for $j = i+2, i+3, \dots, n-1$, and that $\Theta_i(2/a_{2n,i+1}) = 4/(a_{2n,i}a_{2n,i+1})$, from the explicit definition of Θ_i (3.12). Finally, acting first with $\Theta_{n-1} - 2/a_{2n,n-1}$ on Ψ_{π_0} , we see that the only terms not symmetric under $z_{n-1} \leftrightarrow z_n$, hence that do not commute with this action, are $a_{n-1,n}a_{n,2n-1}a_{2n,n-1}$, and from the explicit expression (3.12), we get

$$\left(\Theta_{n-1} - \frac{2}{a_{2n,n-1}}\right)(a_{n-1,n}a_{n,2n-1}a_{2n,n-1}) = a_{n-1,n}a_{2n,n}a_{2n-1,n-1} \quad (3.26)$$

The same reasoning then applies to the resulting function, whose only term non-symmetric under $z_{n-2} \leftrightarrow z_{n-1}$ is $a_{n-2,n-1}a_{2n,n-2}a_{n-1,2n-2}$; at the j^{th} step we act with $\Theta_{n-j} - 2/a_{2n,n-j}$ on the product $a_{n-j,n-j+1}a_{2n,n-j}a_{n-j+1,2n-j}$ with the result $a_{n-j,n-j+1}a_{2n,n-j+1}a_{2n-j,n-j}$, and this goes on until $j = n-1$. Collecting all the terms, we finally get Eq. (3.24).

We conclude that the Ψ_n constructed above, normalized by (3.19), satisfies all equations (3.16), and therefore also satisfies (2.8). Moreover, any GCD of the Ψ_π would divide Ψ_{π_0} of (3.19), but all the factors present in (3.19) are required by the proposition 1, hence this GCD must be a constant and the components of Ψ_n are indeed coprime. Furthermore, due to the structure of Θ_i , we find that the entries Ψ_π are polynomials with total degree and partial degrees in each z_i bounded respectively by the total degree $2n(n-1)$ and partial degrees $2(n-1)$ of Ψ_{π_0} . In fact, we have equality of degrees, and an explicit expression for the highest degree terms in Ψ_π , as the following lemma shows:

Lemma 2. *The terms of Ψ_π of maximal degree read*

$$\Psi_\pi^{\max}(z_1, \dots, z_{2n}) = (-1)^{c(\pi)} \left(\prod_{1 \leq i < j \leq 2n} (z_i - z_j) \right) \prod_{i < j: \pi(i)=j} \frac{1}{z_i - z_j} \quad (3.27)$$

where $c(\pi)$ is the number of crossings of π .

This is proved by induction on the quantity $n(n-1)/2 - c(\pi)$, starting from $\pi = \pi_0$, whose leading degree terms read

$$\Psi_{\pi_0}^{\max}(z_1, \dots, z_{2n}) = (-1)^{n(n-1)/2} \Delta(z) \prod_{i=1}^n \frac{1}{z_i - z_{i+n}} \quad (3.28)$$

where we used the standard notation $\Delta(z) = \prod_{1 \leq i < j \leq 2n} (z_i - z_j)$ for the Vandermonde determinant. This leading term matches (3.27) upon noting that $c(\pi_0) = n(n-1)/2$.

Let us now prove that if π has no little arch linking $(i, i + 1)$, then at leading order in the z , the action of Θ_i preserves the form (3.27), with π replaced by $f_i \cdot \pi$. Assuming that $\pi(i) = k$ and $\pi(i + 1) = \ell$, we see that the product (3.27) may be rewritten as $\Psi_\pi^{\max} = (z_i - z_\ell)(z_{i+1} - z_k)(z_i - z_{i+1})\Phi_\pi$, where the polynomial Φ_π is symmetric under the interchange $z_i \leftrightarrow z_{i+1}$. As any polynomial with such a symmetry may be factored out of the action of Θ_i (the latter affects only non-symmetric terms), we are left with the task of finding the leading behavior at large z_i of

$$\Theta_i(z_i - z_\ell)(z_{i+1} - z_k)(z_i - z_{i+1}) \sim -(z_i - z_k)(z_{i+1} - z_\ell)(z_i - z_{i+1}) \quad (3.29)$$

obtained as an immediate consequence of (3.13), as ∂_i decreases the degree strictly, and only the $-\tau_i$ term contributes at leading order. This proves that $\Psi_{f_i \cdot \pi}^{\max}$ also has the form (3.27), the overall minus sign accounting for the decrease by 1 of the number of crossings. This completes the proof of (3.27) for all entries of Ψ_n , and of Theorem 2. \square

As a side remark, an immediate consequence of Lemma 2 is the property that the maximal degree terms in the ground state vector entries, Ψ_π^{\max} , are invariant under the interchange $z_i \leftrightarrow z_{i+1}$ whenever π has a little arch joining $(i, i + 1)$. Indeed, as an arch joins $(i, i + 1)$, the only term involving z_i to be divided out of the Vandermonde determinant in (3.27) is the skew-symmetric term $(z_i - z_{i+1})$. This leaves us with a manifestly invariant product, and shows that the action of the operators Δ_i on entries of Ψ_n with a little arch joining $(i, i + 1)$ does not increase the degree, as announced.

We conclude this section by stressing that we have explicitly constructed the entries of Ψ by repeated action of the Θ_i on Ψ_{π_0} of (3.19). It is clear that all the entries of Ψ thus constructed are polynomials of the z_i with *integer* coefficients. In particular, in the homogeneous limit where all the z_i tend to the same value (say t), we see that $\Psi_{\pi_0} \rightarrow 1$, and we end up with a vector whose entries are all integers, moreover positive, as Ψ then coincides with the Perron–Frobenius eigenvector of the Hamiltonian of [6], with the same normalization condition. This yields a proof that, in the homogeneous case, once Ψ is normalized by $\Psi_{\pi_0} = 1$, all its entries are positive integers.

4. Sector of permutations

4.1. Definition of the sector P_n and of the associated varieties

A subset P_n of CP_n consists of the $n!$ “permutation patterns” that connect the points $\{1, 2, \dots, n\}$ to the points $\{n+1, n+2, \dots, 2n\}$. Such patterns $\pi \in P_n$ are in one-to-one correspondence with permutations $\hat{\pi} \in \mathcal{S}_n$ of $\{1, \dots, n\}$ via $\pi(i) = \hat{\pi}(n+1-i) + n$, $i = 1, \dots, n$.

To each pattern $\pi \in P_n$ is naturally associated a homogeneous affine variety V_π : following Knutson [9], we consider the three conditions on pairs of $n \times n$ complex matrices X and Y :

- (1) XY lower triangular, YX upper triangular.
- (2) $(XY)_{ii} = (YX)_{\hat{\pi}(i)\hat{\pi}(i)}$ for $i = 1, \dots, n$.
- (3) The matrix Schubert variety conditions: the rank of any upper left (resp. lower right) rectangular submatrix of X (resp. Y) is less or equal to the rank of the corresponding submatrix of the permutation matrix $\hat{\pi}$ (resp. $\hat{\pi}^{-1}$).

The V_π all have the same dimension $n(n+1)$, or equivalently codimension $n(n-1)$ in $M_n(\mathbb{C})^2$. They are conjectured (Sect. 3 of [9]) to be the irreducible components of V which is the set of pairs (X, Y) that satisfy condition (1) only. Note that we use the “lower-upper scheme” here, as in Sect. 1 of [9], where V is denoted by D^0 ; however, starting with Sect. 2, [9] uses the “upper-upper scheme”, hence slightly differing conventions.

4.2. Refined de Gier–Nienhuis conjectures

We first provide the following interpretation of the entries Ψ_π with $\pi \in P_n$ for a special choice of inhomogeneities, which refines a conjecture of Nienhuis and de Gier [6]:

Conjecture 1. Set $z_i = B/(A+B)$, for $i = 1, \dots, n$ and $z_i = A/(A+B)$ for $i = n+1, \dots, 2n$. Then we have the following identification for $\pi \in P_n$:

$$\left(\frac{A+B}{2}\right)^{n(n-1)} \Psi_\pi = d_\pi(A, B) \quad (4.1)$$

where $d_\pi(A, B)$ is the bidegree of V_π which is related to the separate scaling of the matrices X and Y (such a bidegree is defined in Sect. 2.2 of [9]³).

The cases $n = 3$ and $n = 4$ are given in appendix C.

³ Note that we have slightly altered the notation: in [9] the index σ of $d_\sigma(A, B)$ refers to the permutation such that $\pi(i) = n + \sigma(i)$, $i = 1, 2, \dots, n$.

For $A = B = 1$, all z_i are equal, and our conjecture reduces to that of [6]. More precisely, Eq. (4.1) then expresses identities between on the left hand side the entry of the homogeneous problem (i.e. of the null vector of the Hamiltonian H_n), and on the right hand side the usual degrees of the algebraic varieties.

In order to go further, we note that for any $\pi \in P_n$, the entry Ψ_π has no little arch connecting pairs (i, j) with $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq 2n$. This motivates the following redefinitions: set $p_i = z_{n+1-i}$, $q_i = z_{n+i}$, $i = 1, \dots, n$ and

$$\Psi_\pi(z_1, \dots, z_{2n}) = \prod_{1 \leq i < j \leq n} (1 + z_i - z_j) \prod_{n+1 \leq k < l \leq 2n} (1 + z_k - z_l) \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) \quad (4.2)$$

where δ_π is some polynomial of its $2n$ variables.

To interpret δ_π , we also need a more general action of a torus of dimension $2n+2$ that maps (X, Y) to $(aPXQ^{-1}, bQYP^{-1})$; it allows to define multidegrees, i.e. torus-equivariant cohomology. With respect to this action, the weights of the variables are:

$$[X_{ij}] = A + p_i - q_j \quad [Y_{ij}] = B + q_i - p_j \quad (4.3)$$

We now have the following stronger conjecture:

Conjecture 2. $\delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n)$ is the $(2n+2)$ -multidegree of V_π associated to the weights (4.3) in which one sets $A = B = 1$.

That Conjecture 2 implies Conjecture 1 is due to the following simple scaling property: if one sets $A = B = 1$ in Eq. (4.3), and performs the change of variables

$$p_i = \frac{A' + 2p'_i}{A' + B'} \quad q_i = \frac{B' + 2q'_i}{A' + B'} \quad (4.4)$$

one finds $[X_{ij}] = \frac{2}{A'+B'}(A' + p'_i - q'_j)B$ and $[Y_{ij}] = \frac{2}{A'+B'}(B' + q'_i - p'_j)$, which are identical to the original weights up to a factor $2/(A' + B')$. Since the multidegrees are homogeneous polynomials of degree $n(n-1)$ (the codimension of V_π), the full $(2n+2)$ -multidegree can be extracted from δ_π by simply pulling out the factor $(2/(A' + B'))^{n(n-1)}$. As a special case, if we set all $p'_i = q'_i = 0$, we recover Eq. (4.1).

Note also that the terms of highest degree of δ_π are now interpreted, using the same scaling argument, as the $2n$ -multidegree of V_π associated to the weights (4.3) with $A = B = 0$. Explicitly,

$$\delta_\pi^{\max}(p_1, \dots, p_n, q_1, \dots, q_n) = (-1)^{c(\pi)} \prod_{\substack{i,j=1 \\ j \neq \hat{\pi}(i)}}^n (p_i - q_j) \quad (4.5)$$

In Sect. 4.5, we shall give a simple and efficient way to compute the δ_π for arbitrary n . But first we show some simple properties satisfied by the entries Ψ_π , $\pi \in P_n$; these properties have clear meaning for the multidegrees (see also Sect. 2.2 of [9] for analogous statements on (bi)degrees). As a simple example, the behavior of Ψ_n under rotation and reflection (Eqs. (2.9)–(2.10)) implies, modulo Conjecture 1, the third and fourth statements in Prop. 4 of [9]; these statements could be easily extended to multidegrees, and indeed, we find for the conjectured multidegrees δ_π :

$$\delta_{\rho^n \cdot \pi}(q_n, \dots, q_1, p_n, \dots, p_1) = \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) \quad (4.6a)$$

$$\delta_{r \cdot \pi}(-q_1, \dots, -q_n, -p_1, \dots, -p_n) = \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) \quad (4.6b)$$

$$\delta_{r' \cdot \pi}(-p_n, \dots, -p_1, -q_n, \dots, -q_1) = \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) \quad (4.6c)$$

where $\widehat{\rho^n \cdot \pi} = \hat{\pi}_0 \hat{\pi}^{-1} \hat{\pi}_0$, $\widehat{r \cdot \pi} = \hat{\pi}^{-1}$ and $\widehat{r' \cdot \pi} = \hat{\pi}_0 \hat{\pi} \hat{\pi}_0$.

4.3. Sum rule for the permutation sector P_n

Theorem 3. *The sum of entries of Ψ_n corresponding to permutation patterns has the following factorized form:*

$$\begin{aligned} Y_n(z_1, \dots, z_{2n}) &= \sum_{\pi \in P_n} \Psi_\pi(z_1, z_2, \dots, z_{2n}) \\ &= \left(\prod_{1 \leq i < j \leq n} (1 + z_i - z_j)(2 - z_i + z_j) \right) \left(\prod_{n+1 \leq k < l \leq 2n} (1 + z_k - z_l)(2 - z_k + z_l) \right) \end{aligned} \quad (4.7)$$

Proof. Let us introduce the linear form b_n that is the characteristic function of P_n , namely with entries in the canonical basis

$$b_\pi = \begin{cases} 1 & \text{if } \pi \in P_n \\ 0 & \text{otherwise} \end{cases} \quad (4.8)$$

b_n satisfies

$$b_n I = b_n, \quad b_n f_i = b_n, \quad b_n e_i = 0, \quad i \neq n, 2n \quad (4.9)$$

as no little arch may connect points among $\{1, 2, \dots, n\}$ or $\{n+1, n+2, \dots, 2n\}$ in $\pi \in P_n$. We now act with the characteristic function b_n of the permutation sector P_n (4.8) on the matrix \check{R} . Using the relations (4.9), we immediately find

$$b_n \check{R}_i(z, w) = \frac{(1 + \frac{1}{2}(w - z))(1 + z - w)}{(1 - \frac{1}{2}(w - z))(1 + w - z)} b_n, \quad i \neq n, 2n \quad (4.10)$$

Noting finally that $Y_n = b_n \cdot \Psi_n$, and taking the scalar product of b_n with (3.7), we find that the r.h.s. of (4.7) divides the sum on the l.h.s. Moreover, the degree of the r.h.s. is $2n(n-1)$, the same as that of the sum Y_n by application of Theorem 2. The two terms must therefore be proportional by a constant, further fixed to be 1 by considering the terms of maximal degree ($2n(n-1)$) in Y_n . Indeed, the maximal degree term in the r.h.s. of (4.7) reads

$$\Delta^2(z_1, \dots, z_n) \Delta^2(z_{n+1}, \dots, z_{2n}) \quad (4.11)$$

where Δ stands for the Vandermonde determinant. On the other hand, resumming all leading behaviors (3.27) over the permutation patterns $\pi \in P_n$, we get the following quantity

$$\begin{aligned} Y_n^{max}(z_1, \dots, z_{2n}) &= \Delta(z_1, \dots, z_{2n}) \sum_{\pi \in P_n} (-1)^{c(\pi)} \prod_{\substack{\text{pairs } (i,j) \\ j=\pi(i), \ i=1,2,\dots,n}} \frac{1}{z_i - z_j} \\ &= \Delta(z_1, \dots, z_{2n}) \det \left(\frac{1}{z_i - z_{n+j}} \right)_{1 \leq i, j \leq n} \end{aligned} \quad (4.12)$$

where we have identified the sign $(-1)^{c(\pi)}$ with the signature of the underlying permutation, eventually leading to the determinant. The identity between (4.11) and (4.12) is just the Cauchy determinant formula:

$$\frac{\Delta(z_1, \dots, z_n) \Delta(z_{n+1}, \dots, z_{2n})}{\prod_{i,j=1}^n (z_i - z_{n+j})} = \det \left(\frac{1}{z_i - z_{n+j}} \right)_{1 \leq i, j \leq n} \quad (4.13)$$

□

Note that the sum rule (4.7) translates, at the special values of inhomogeneities considered in Sect. 4.1 for the Conjecture 1, into the following

$$\sum_{\pi \in P_n} d_\pi(A, B) = \left(\frac{A+B}{2} \right)^{n(n-1)} Y_n \left(\frac{B}{A+B}, \dots, \frac{A}{A+B}, \dots \right) = (A+B)^{n(n-1)} \quad (4.14)$$

which is in agreement with the first statement in Prop. 4 of [9]. Yet another expression, equivalent to Theorem 3, is:

$$\sum_{\pi \in P_n} \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) = \prod_{1 \leq i < j \leq n} (2 + p_i - p_j)(2 - q_i + q_j) \quad (4.15)$$

which is clearly the multidegree of the whole variety V (equations enforcing condition (1)).

Remark: rotated versions of the sum rule (4.7) are also available, upon using the general cyclic covariance property of Ψ_n (2.9), namely involving $Y_n^{(i)} = Y_n(z_k \rightarrow z_{k+i})$, with $Y_n^{(0)} \equiv Y_n$. These correspond to sums of entries of Ψ_n over the rotated permutation pattern sets $P_n^{(i)}$ that connect the points $\{i+1, i+2, \dots, i+n\}$ to $\{n+i+1, n+i+2, \dots, i\}$ and will be used in Sect. 5.1 (proof of Theorem 4).

4.4. Factorization in the permutation sector P_n

We may prove a general factorization property for the fully decomposable permutation patterns $\pi \in P_n$ defined as follows. Assume the points $1, 2, \dots, n$ are partitioned into two sets $R_1 = \{1, 2, \dots, r\}$, $R_2 = \{r+1, r+2, \dots, n\}$; define also $S_1 = R_1 + n$, $S_2 = R_2 + n$. A permutation pattern $\pi \in P_n$ is called fully decomposable with respect to the partition (R_1, R_2) if π only connects the points of R_1 to those of S_1 and the points of R_2 to those of S_2 . We denote by π_i the restriction of π to the set $R_i \cup S_i$.

Proposition 2. *For any fully decomposable π , we have the factorization property*

$$\Psi_\pi(z_1, \dots, z_{2n}) = \prod_{(i,j) \in X} (1 + z_i - z_j) \Psi_{\pi_1}(z_{R_1 \cup S_1}) \Psi_{\pi_2}(z_{R_2 \cup S_2}) \quad (4.16)$$

where $X = R_1 \times R_2 \cup R_2 \times S_1 \cup S_1 \times S_2 \cup S_2 \times R_1$, and z_I denotes the sequence of $z_i, i \in I$.

To prove (4.16), we proceed by induction on $n(n-1)/2 - c(\pi)$. we start from the link pattern π_0 , viewed as a fully decomposable pattern with respect to (R_1, R_2) . From the explicit expression (3.19), we immediately get (4.16) upon noting that the restrictions π_1 and π_2 are themselves the maximally crossing patterns π_0 in their respective sets P_r and P_{n-r} . Assume some fully decomposable π satisfies (4.16). We may reduce by 1 the number of crossings of π by acting on π with some f_i , with either $i \in \{1, 2, \dots, r-1\}$ or $i \in \{r+1, r+2, \dots, n-1\}$. The corresponding entry $\Psi_{f_i \cdot \pi}$ is obtained by acting with Θ_i on Ψ_π . But this action only affects the corresponding restriction π_1 or π_2 , within which the uncrossing is performed. Hence the form (4.16) is preserved, and we simply have to substitute $\pi \rightarrow f_i \cdot \pi$ and either $\pi_1 \rightarrow f_i \cdot \pi_1$ or $\pi_2 \rightarrow f_i \cdot \pi_2$. This completes the proof of (4.16). \square

As a corollary, when all z_i are taken to zero, Eq. (4.16) translates into a factorization property conjectured in [6]. The corresponding property for bidegrees is the second statement of Prop. 4 of [9]. More generally, we can rewrite it in terms of our conjectured multidegrees:

$$\begin{aligned} \delta_\pi(p_1, \dots, p_n, q_1, \dots, q_n) &= \prod_{i=1}^{n-r} \prod_{j=1}^r (1 + p_i - q_j) \prod_{i=r+1}^n \prod_{j=n-r+1}^n (1 + q_i - p_j) \\ &\quad \delta_{\pi_1}(p_{n-r+1}, \dots, p_n, q_1, \dots, q_r) \delta_{\pi_2}(p_1, \dots, p_{n-r}, q_{r+1}, \dots, q_n) \end{aligned} \quad (4.17)$$

The factors $\prod_{i=1}^{n-r} \prod_{j=1}^r (1 + p_i - q_j) \prod_{i=r+1}^n \prod_{j=n-r+1}^n (1 + q_i - p_j)$ correspond to equations enforcing the block-triangular shape of the matrices X and Y for a decomposable π .

4.5. Sketch of proof of the multi-parameter de Gier–Nienhuis conjecture

It is not the purpose of the present paper to give a rigorous proof of Conjecture 2. However, we shall indicate the main steps of the proof, leaving aside various technicalities. Note that no results in this paper depend on proving Conjecture 2.

The property to be proved is written in short as: $\deg V_\pi = \delta_\pi$ (deg denoting the multidegree as in Conjecture 2, i.e. with $A = B = 1$). The proof proceeds as usual by induction on the number of crossings of π .

★ For the case of the long permutation $\hat{\pi}_0$, we have the explicit formula (3.19), or with our redefinitions:

$$\delta_{\pi_0}(p_1, \dots, p_n, q_1, \dots, q_n) = \prod_{\substack{1 \leq i, j \leq n \\ i+j < n+1}} (1 + p_i - q_j) \prod_{\substack{1 \leq i, j \leq n \\ i+j > n+1}} (1 + q_i - p_j) \quad (4.18)$$

But the corresponding variety V_{π_0} is known explicitly:

$$V_{\pi_0} = \{(X, Y) \in M_n(\mathbb{C})^2 \mid X \text{ lower right triangular, } Y \text{ upper left triangular}\} \quad (4.19)$$

The equations defining V_{π_0} are simply: $X_{ij} = 0$ for $i + j < n + 1$ and $Y_{ij} = 0$ for $i + j > n + 1$; using Eq. (4.3) with $A = B = 1$, we immediately find that $\deg V_{\pi_0}$ equals the r.h.s. of Eq. (4.18).

★ Induction. We want to show the property for a certain permutation pattern, assuming it is true for all permutation patterns with higher number of crossings. We can always write this pattern as $f_{\bar{i}} \cdot \pi$, $\bar{i} = 1, \dots, n - 1$, such that π has one more crossing than $f_{\bar{i}} \cdot \pi$. In terms of permutations, $\widehat{f_{\bar{i}} \cdot \pi} = \hat{\pi} \tau_i$ with $i = n + 1 - \bar{i}$, that is $f_{\bar{i}}$ acts (as elementary transposition τ_i) by multiplication on the right.⁴

The crucial point is once more to use the formula (3.11) which relates $\Psi_{f_{\bar{i}} \cdot \pi}$ to Ψ_π via the operator Θ_j defined by Eq. (3.12). After taking into account our redefinition (4.2), we find the much simpler expression

$$\delta_{f_{\bar{i}} \cdot \pi} = (2\partial_i - \tau_i) \delta_\pi \quad (4.20)$$

⁴ Note that one could have used a $f_{\bar{i}}$ with $\bar{i} = n + 1, \dots, 2n - 1$; this would correspond to multiplication of the permutation $\hat{\pi}$ on the left. In our multidegree setting (similar to double Schubert polynomials) there is complete symmetry between left and right multiplication, corresponding to operators acting on the p_i or on the q_i .

following immediately from (3.13). Here τ_i is the natural action of the symmetric group that exchanges variables p_i and p_{i+1} , and ∂_i is the *divided difference* operator $\partial_i = \frac{1}{p_{i+1}-p_i}(\tau_i - 1)$ (with the usual sign convention). Note that this is a known representation of the symmetric group, studied for example in [11].

It is better to rewrite Eq. (4.20) as:

$$\delta_{f_i \cdot \pi} + \delta_\pi = (2 + p_i - p_{i+1})\partial_i \delta_\pi \quad (4.21)$$

Roughly speaking, ∂_i corresponds to the action of the elementary transposition τ_i in the matrix Schubert variety conditions (3), whereas $2 + p_i - p_{i+1}$ is the additional equation $(XY)_{i+1} = 0$ (part of condition (1)) which was lost in the process.

More explicitly, call X_i the row vectors of X and Y_i the column vectors of Y . For any variety W inside $M_n(\mathbb{C})^2$ with a torus action, define $\partial_i W \equiv \{(X', Y') | X'_j = X_j \ \forall j \neq i, X'_i = X_i + uX_{i+1}, Y'_j = Y_j \ \forall j \neq i+1, Y'_{i+1} = Y_{i+1} - uY_i \text{ for } (X, Y) \in W, u \in \mathbb{C}\}$. One can show that $W \mapsto \partial_i W$ translates into the operator ∂_i for multidegrees (indeed it decreases codimension = degree by 1, and clearly does not act on variables other than p_i, p_{i+1} or on symmetric functions of p_i, p_{i+1} ; all these properties characterize ∂_i up to normalization, which is easily fixed).

Now apply this operation ∂_i to the variety V_π . By direct inspection, $\partial_i V_\pi$ satisfies condition (3) in which $\hat{\pi}$ is replaced with $\hat{\pi}\tau_i$ (standard reasoning for matrix Schubert varieties), as well as the set of equations defining condition (1) (noting that $Y'X' = YX$) except for $(X'Y')_{i+1} = \langle X'_i | Y'_{i+1} \rangle = 0$. We therefore compute $\langle X'_i | Y'_{i+1} \rangle = u(\langle X_{i+1} | Y_{i+1} \rangle - \langle X_i | Y_i \rangle - u \langle X_{i+1} | Y_i \rangle)$, and find that the equation $\langle X'_i | Y'_{i+1} \rangle = 0$ nicely factorizes into $u = 0$, which of course defines V_π , and a non-trivial linear equation for u :

$$\langle X_{i+1} | Y_{i+1} \rangle - \langle X_i | Y_i \rangle - u \langle X_{i+1} | Y_i \rangle = 0 \quad (4.22)$$

Now we want to check condition (2) when this equation is satisfied. We have $(Y'X')_{jj} = (YX)_{jj}$ for all j , and $(X'Y')_{jj} = (XY)_{jj}$ for $j \neq i, i+1$. Furthermore,

$$(X'Y')_{ii} = \langle X'_i | Y'_i \rangle = \langle X_i | Y_i \rangle + u \langle X_{i+1} | Y_i \rangle = \langle X_{i+1} | Y_{i+1} \rangle = (XY)_{i+1 i+1} \quad (4.23a)$$

$$(X'Y')_{i+1 i+1} = \langle X'_{i+1} | Y'_{i+1} \rangle = \langle X_{i+1} | Y_{i+1} \rangle - u \langle X_{i+1} | Y_i \rangle = \langle X_i | Y_i \rangle = (XY)_{ii} \quad (4.23b)$$

using Eq. (4.22). We conclude that (X', Y') satisfies condition (2) in which $\hat{\pi}$ is replaced with $\hat{\pi}\tau_i$. Since the operation is an involution, all of $V_{f_i \cdot \pi}$ is obtained this way.

What we have found is that the additional relation $(X'Y')_{ii+1} = 0$ restricts $\partial_i V_\pi$ to $V_\pi \cup V_{f_{\bar{i}} \cdot \pi}$. This means that it increases codimension by 1, i.e. is independent from other equations and therefore amounts to multiplication by $2 + p_i - p_{i+1}$ of the multidegree (cf Eq. (4.3) with $A = B = 1$). This proves that $\deg V_{f_{\bar{i}} \cdot \pi} + \deg V_\pi = (2 + p_i - p_{i+1}) \partial_i \deg V_\pi$; by the induction hypothesis $\deg V_\pi = \delta_\pi$, and comparing with Eq. (4.21), we conclude that $\deg V_{f_{\bar{i}} \cdot \pi} = \delta_{f_{\bar{i}} \cdot \pi}$. \square

As already mentioned at the end of Sect. 3, a corollary of the recursive construction of the Ψ_π , or equivalently of the δ_π in the permutation sector, is that they are sums of products of $1 - z_i + z_j$, and in particular polynomials with integer coefficients; this could also be seen as a consequence, assuming Conjecture 2, of the general theorem 1.7.1. of [12].

5. Recursion relations and full sum rule

In this section, we will derive recursion relations relating specialized entries of Ψ_n to entries of Ψ_{n-1} , that will allow us to prove the full sum rule.

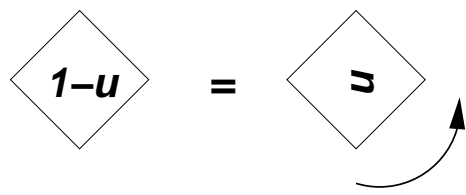
5.1. Recursion relations for the entries of Ψ_n

Let us examine the case when a little arch connects two neighboring points $i, i+1$ in some $\pi \in CP_n$. We have the following

Lemma 3. *Let φ_i denote the embedding $CP_{n-1} \rightarrow CP_n$ that inserts a little arch between positions $i-1$ and i and relabels the later positions $j \rightarrow j+2$ in any $\pi \in CP_{n-1}$; we also denote by φ_i the induced embedding of vector spaces. We have*

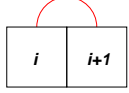
$$\begin{aligned} T_n(t, z_1, \dots, z_i, z_{i+1} = z_i + 1, \dots, z_{2n}) \varphi_i \\ = \frac{1}{4} (t - z_i)(1 - z_i + t)(2 - t + z_i)(3 - t + z_i) \varphi_i T_{n-1}(z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_{2n}) \end{aligned} \quad (5.1)$$

Proof. This is a direct consequence of the unitarity relation (3.2) together with the so-called crossing relation



$$\diamond(1-u) = \diamond(u) \quad (5.2)$$

in which the second picture has been rotated by $+\pi/2$. Indeed, such a rotation exchanges the roles of I and e_i while f_i is left invariant, and the coefficients in (2.3) are interchanged accordingly under $u \rightarrow 1 - u$. However, it is more instructive to prove (5.1) directly, by attaching to the little arch the two plaquettes at sites i and $i+1$. We denote by $u_i = 1 - t + z_i$, $v_i = \frac{1}{2}(t - z_i)(1 - t + z_i)$ and $w_i = t - z_i$. Pictorially, this gives rise to 9 situations:



$$\begin{aligned}
&= w_i w_{i+1} \text{ (diagram)} + u_i u_{i+1} \text{ (diagram)} + u_i w_{i+1} \text{ (diagram)} + u_i v_{i+1} \text{ (diagram)} + v_i w_{i+1} \text{ (diagram)} \\
&+ w_i v_{i+1} \text{ (diagram)} + v_i u_{i+1} \text{ (diagram)} \\
&+ w_i u_{i+1} \text{ (diagram)} + v_i v_{i+1} \text{ (diagram)}
\end{aligned} \tag{5.3}$$

We have displayed on the same line the terms contributing to the same picture. We now note that when $z_{i+1} = z_i + 1$, the first and second line both have a global vanishing factor $w_i w_{i+1} + u_i u_{i+1} + u_i w_{i+1} + u_i v_{i+1} + v_i w_{i+1} = w_i v_{i+1} + v_i u_{i+1} = 0$. We are only left with the contribution where the little arch has safely gone across the horizontal line, and where in passing the two spaces i and $i + 1$ have been erased. The prefactor is

$$w_i u_{i+1} + v_i v_{i+1} = \frac{1}{4}(t - z_i)(1 - z_i + t)(2 - t + z_i)(3 - t + z_i) \tag{5.4}$$

and yields the prefactor in Eq. (5.1). \square

Together with the results of previous sections, this leads to:

Theorem 4. *For a given $\pi \in CP_n$, taking $z_{i+1} = z_i + 1$, we have either of the two following situations:*

(i) *There is no little arch $(i, i + 1)$ in π . Then*

$$\Psi_\pi(z_1, \dots, z_i, z_{i+1} = z_i + 1, \dots, z_{2n}) = 0 \tag{5.5}$$

(ii) *There is a little arch connecting i and $i + 1$. Then we have the recursion relation*

$$\begin{aligned}
&\Psi_\pi(z_1, \dots, z_i, z_{i+1} = z_i + 1, \dots, z_{2n}) = \\
&\left(\prod_{\substack{k=1 \\ k \neq i, i+1}}^{2n} (1 + z_{i+1} - z_k)(1 + z_k - z_i) \right) \Psi_{\pi'}(z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_{2n}) .
\end{aligned} \tag{5.6}$$

between the entry Ψ_π of Ψ_n and the entry $\Psi_{\pi'}$ of Ψ_{n-1} , where π' is the link pattern π with the little arch $i, i+1$ removed ($\pi = \varphi_i \pi'$, $\pi' \in CP_{n-1}$).

Proof. The situation (i) is covered by Proposition 1. To prove (5.6), we use Lemma 3, and act on Ψ_{n-1} , with the obvious values of the parameters. We have $T\varphi_i \Psi_{n-1} = \Lambda/\Lambda' \varphi_i T' \Psi_{n-1} = \Lambda \varphi_i \Psi_{n-1}$, hence $\varphi_i \Psi_{n-1}$ is proportional to Ψ_n , when $z_{i+1} = z_i + 1$. This yields Eq. (5.6), up to a proportionality factor say $\gamma_{n,i}$, rational fraction of the parameters. This factor is further fixed by applying (5.5)–(5.6) to the sum over the suitably rotated set of permutation patterns $P_n^{(i)}$, that allows for little arches in positions $(i, i+1)$ or $(i+n, i+n+1)$. Due to the properties (i-ii), the only non-vanishing contributions to this sum of entries when $z_{i+1} = z_i + 1$ are those for which a little arch connects $(i, i+1)$. These are the images under φ_i of the permutation sector $P_{n-1}^{(i-1)}$, hence we get

$$Y_n^{(i)}(z_1, \dots, z_{2n}) \Big|_{z_{i+1}=z_i+1} = \gamma_{n,i} Y_{n-1}^{(i-1)}(z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_{2n}) \quad (5.7)$$

Applying the suitable rotations to the result of Theorem 3 yields the value of $\gamma_{n,i}$ and (5.6) follows. \square

5.2. Sum rule for the entries

We now compute the sum of all the entries of Ψ_n and express it in terms of a Pfaffian. We start with the following

Lemma 4. *The sum of entries of Ψ_n :*

$$Z_n(z_1, \dots, z_{2n}) = \sum_{\pi \in CP_n} \Psi_\pi(z_1, \dots, z_{2n}) \quad (5.8)$$

is a symmetric polynomial.

Proof. the linear form v_n , with entries $v_\pi = 1$, clearly satisfies $v_n I = v_n$, $v_n f_i = v_n$, $v_n e_i = v_n$, as each of the operators I, f_i, e_i sends each link pattern to a unique link pattern. Therefore $v_n \check{R}_i(z, w) = v_n$. The sum over entries is nothing but $Z_n = v_n \cdot \Psi_n$, and taking the scalar product of v_n with Eq. (3.7), we deduce that Z_n is invariant under the interchange of $z_i \leftrightarrow z_{i+1}$, hence is fully symmetric. \square

By virtue of Theorem 2 and Lemma 4, $Z_n(z_1, \dots, z_{2n})$, the sum of entries of Ψ_n , is a symmetric polynomial of total degree $2n(n-1)$ and partial degree $2n-2$ in each variable, subject to the recursion relation obtained by summing Eqs. (5.5)–(5.6) over all $\pi \in CP_n$:

$$\begin{aligned} & Z_n(z_1, \dots, z_{2n})|_{z_{i+1}=z_i+1} \\ &= \left(\prod_{\substack{k=1 \\ k \neq i, i+1}}^{2n} (1 + z_{i+1} - z_k)(1 + z_k - z_i) \right) Z_{n-1}(z_1, \dots, z_{i-1}, z_{i+2}, \dots, z_{2n}) \end{aligned} \quad (5.9)$$

Together with the initial condition $Z_1(z_1, z_2) = 1$, these properties determine it completely and allow us to prove the

Theorem 5. *The sum of all entries Z_n has a Pfaffian formulation:*

$$Z_n(z_1, \dots, z_{2n}) = \text{Pf} \left(\frac{z_i - z_j}{1 - (z_i - z_j)^2} \right)_{1 \leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} \frac{1 - (z_i - z_j)^2}{z_i - z_j} \quad (5.10)$$

Proof. The r.h.s. of (5.10), which we denote by K_n , is a symmetric polynomial of the z_i , with total degree $2n(n-1)$ and partial degree $2(n-1)$ in each variable, and such that when say $z_2 \rightarrow z_1 + 1$, the Pfaffian degenerates into $(z_1 - z_2)/(1 - (z_1 - z_2)^2)$ times that for z_3, z_4, \dots, z_{2n} , while the other quantity reduces to $(1 - (z_1 - z_2)^2)/(z_1 - z_2)$ times $\prod_{3 \leq j \leq 2n} (1 + z_2 - z_j)(1 + z_j - z_1)$ times that for z_3, z_4, \dots, z_{2n} . As the leading singularity is cancelled, this gives the recursion relation

$$K_n(z_1, \dots, z_{2n})|_{z_2=z_1+1} = \left(\prod_{j=3}^{2n} (1 + z_2 - z_j)(1 + z_j - z_1) \right) K_{n-1}(z_3, \dots, z_{2n}) \quad (5.11)$$

This is nothing but (5.9) for $i = 1$, and we deduce that $Z_n = \alpha K_n$ for some numerical factor α , fixed to be 1 by the initial condition $Z_1(z_1, z_2) = 1$.

Note that the Pfaffian is naturally a sum over pairings of $2n$ objects, indexed equivalently by crossing link patterns, and weighted by a fermionic sign factor. This decomposes naturally Z_n into an alternating sum over link patterns. In particular taking all z_i to be large, we obtain the leading degree contribution to Z_n , $Z_n^{max} = \Delta(z) \text{Pf}(1/(z_i - z_j))_{1 \leq i < j \leq 2n}$, which matches exactly the sum of the leading terms (3.27) obtained in Lemma 2 above.

We get an interesting corollary of Theorem 5 by taking all z_i to zero. To do this, notice that if $Z = \text{Pf}(A) / \prod_{i < j} f(z_i - z_j)$, $A_{ij} = f(z_i - z_j)$, f an odd function such that $f'(0) \neq 0$, then when all z_i tend to 0, we have:

$$Z \rightarrow \frac{1}{f'(0)^{n(2n-1)}} \text{Pf} \left(\frac{1}{i!j!} \frac{\partial^i}{\partial z^i} \frac{\partial^j}{\partial w^j} f(z-w) \Big|_{z=w=0} \right)_{0 \leq i, j \leq 2n-1} \quad (5.12)$$

With $f(x) = x/(1-x^2)$, this gives

$$Z_n(0, \dots, 0) = \text{Pf} \left[(-1)^j \binom{i+j}{i} \delta_{i+j, 1[2]} \right]_{0 \leq i, j \leq 2n-1} \quad (5.13)$$

where the Kronecker delta symbol ensures that the matrix element vanishes unless $i+j$ is odd. We also have the following determinantal expression, immediately following from (5.13):

$$Z_n(0, \dots, 0) = \det \left[\binom{2i+2j+1}{2i} \right]_{0 \leq i, j \leq n-1} \quad (5.14)$$

These numbers play for the crossing $O(1)$ loop model the same role as that played for the non-crossing loop model by the numbers A_n of alternating sign matrices of size $n \times n$. They read

$$1, 7, 307, 82977, 137460201, 1392263902567 \dots \quad (5.15)$$

for $n = 1, 2, 3, 4, 5, 6 \dots$. A remark is in order. One possible interpretation of the numbers (5.14), via the Lindström–Gessel–Viennot formula [13], is that they count the total number of n -tuples of non-intersecting lattice paths subject to the following constraints: the paths are drawn on the edges of the square lattice, are non-intersecting, and may only make steps of $(-1, 0)$ (horizontal to the left) or $(0, 1)$ (vertical up), and start at the points $(2i, 0)$, while they end up at the points $(0, 2i+1)$, $i = 0, 1, \dots, n-1$. In (5.14), the combinatorial number $\binom{2i+2j+1}{2i}$ is simply the number of configurations of a single path starting at $(2i, 0)$ and ending at $(0, 2j+1)$.

Finally, it is easy to prove, using standard integrable or matrix model techniques, that the numbers (5.14) behave for large n as $Z_n(0, \dots, 0) \approx \left(\frac{\pi}{2}\right)^{2n^2}$. This is to be compared with the standard asymptotics for the number of $n \times n$ alternating sign matrices $A_n \approx \left(\frac{3\sqrt{3}}{4}\right)^{n^2}$.

6. Conclusion

In this paper, we have investigated the ground state vector Ψ_n of the inhomogeneous $O(1)$ crossing loop model on a cylinder of perimeter $2n$. The inhomogeneities translate into a collection of $2n$ spectral parameters z_1, z_2, \dots, z_{2n} , in terms of which we have proved that, when suitably normalized, all entries of Ψ_n are polynomials of total degree $2n(n-1)$. By further characterizing these entries, using in particular an algorithm for generating them recursively, we have been able to find compact formulas for their sum or partial sum over a subset of entries related to permutations of n objects.

6.1. Comparison with the non-crossing loop case

Comparing this case to that of the standard (non-crossing) $O(1)$ loop model studied in [5], the situation is now much simpler. Indeed, the transitivity of the move generated by f_i on the crossing link patterns has allowed us to describe all the entries of Ψ_n simply in terms of successive local actions of the gauged divided difference operators Θ_i (3.12) on that corresponding to the maximally crossing link pattern. In particular, as Θ_i is degree-preserving, we have had no difficulty in proving that the coefficients of Ψ_n are polynomials of degree $2n(n-1)$, as opposed to the non-crossing case of [5], where no such simple operator is available, and where the degree issue was highly non-trivial, and eventually had to be settled by use of the algebraic Bethe Ansatz. Other properties, such as recursion relations when neighboring spectral parameters become related, turn out to be very similar in the two models. In particular, in light of the formula (5.10) for the sum of all entries of Ψ_n , we have found an analogous Pfaffian formula for the *square* of the Izergin–Korepin/Okada determinant Z_n^{IK} , equal to the sum of entries in the non-crossing case, reading:

$$Z_n^{IK}(z_1, z_2, \dots, z_{2n})^2 = \text{Pf} \left(\frac{z_i - z_j}{z_i^2 + z_i z_j + z_j^2} \right)_{1 \leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} \frac{z_i^2 + z_i z_j + z_j^2}{z_i - z_j} \quad (6.1)$$

The latter corresponds to the partition function of the inhomogeneous non-crossing $O(1)$ loop model on a complete (infinite) cylinder of perimeter $2n$. The analogies between the crossing and non-crossing loop cases lead us to expect the existence of a much more general treatment of integrable lattice models involving loops, which would display these common features. Actually, the right unifying algebraic setting seems to be the so-called “double affine Hecke algebras” [14] which typically mix loop-like operators with the action of the symmetric group on spectral parameters (see also [15]).

6.2. A positive extension of Schubert polynomials

An intriguing and new feature of the crossing loop model is the emergence of a special subset of entries of Ψ_n , indexed by permutations. Roughly speaking, the latter is obtained by projecting out the generators e_i , $i \neq n, 2n$, via the relations (4.9). We have shown that the action of the operators Θ_i drastically simplifies in this sector so as to resemble the standard divided difference operators commonly used in Schubert calculus. In this respect, the conjecture of [6] relating the ground state of the homogeneous crossing loop model to the degrees of some varieties certainly loses much of its mystery. We have actually refined this conjecture so as to include all spectral parameter dependence, by interpreting a suitably normalized version of the entries of Ψ_n in the permutation sector as the multidegree of the varieties studied in [9], and gave a sketch of proof of it. Playing around with changes of variables, we have also found an interesting family of polynomials corresponding to the permutation sector, which seems to have only non-negative integer coefficients. These correspond to the specializations $z_{n+1-i} = (t_i - 1)/(t_i + 1)$, $z_{n+i} = 0$, for $i = 1, 2, \dots, n$. More precisely, we define the family of polynomials s_π via:

$$s_\pi(t_1, t_2, \dots, t_n) = \frac{2^{-n(n-1)} \prod_{i=1}^n (1 + t_i)^{2(n-1)}}{\prod_{1 \leq i < j \leq n} (1 - t_i + 3t_j + t_i t_j)} \Psi_\pi \left(\frac{t_n - 1}{t_n + 1}, \dots, \frac{t_1 - 1}{t_1 + 1}, 0, \dots \right) \quad (6.2)$$

The top member of the family reads simply $s_{\pi_0} = \prod_{1 \leq i \leq n} t_i^{n-i}$, while all other members are obtained by repeated actions of the operators θ_i , $i = 1, 2, \dots, n-1$, defined by

$$\theta_i s_\pi = ((1 + t_i)(1 + t_{i+1})\partial_i - \tau_i) \quad (6.3)$$

(θ_i implementing as usual multiplication on the right by the elementary transposition of the permutation $\hat{\pi}$ such that $\pi(i) = n + \hat{\pi}(n + 1 - i)$). The lowest degree term in the s_π , s_π^{min} , is readily identified with the Schubert polynomial indexed by $\hat{\pi}$. The higher degree terms all turn out experimentally to have non-negative integer coefficients. Eq. (6.3) gives a very efficient way of computing the bidegrees of conjecture 1 (4.1): indeed, the latter are simply recovered by taking $t_i = A/B$ for all i and premultiplying s_π by $B^{n(n-1)}$. This extension of Schubert polynomials awaits some good algebro-geometric or combinatorial interpretation.

6.3. Other entries of Ψ and more combinatorics

This subset of entries of particular interest should not let us forget about the general picture we have obtained. It is actually quite suggestive that for instance the change of variables $z_i = A/(A+B)$, $z_{i+n} = B/(A+B)$ for $i = 1, 2, \dots, n$, together with the global multiplication by $(A+B)^{2n(n-1)-(n-2)-\text{mod}(n,2)}2^{-n(n-1)}$ leads to homogeneous polynomials of A, B for *all* entries of Ψ_n (not just those of the permutation sector), all with *non-negative* integer coefficients. It would be very interesting to find an interpretation for these other entries as well.

More combinatorics are undoubtedly hidden in the $O(1)$ crossing loop model. For instance, the specialization $z_1 = (t-1)/(t+1)$ while all other z_i are taken to zero leads after multiplication by a global factor $((1+t)/2)^{2(n-1)}$ to entries that are all polynomials of t with *non-negative* integer coefficients. The sum over all these entries produces a “refinement” of the numbers (5.15) into polynomials of t , reading up to $n = 4$:

$$\begin{aligned} P_1(t) &= 1, & P_2(t) &= 1 + 5t + t^2, & P_3(t) &= 7 + 63t + 167t^2 + 63t^3 + 7t^4, \\ P_4(t) &= 307 + 3991t + 18899t^2 + 36583t^3 + 18899t^4 + 3991t^5 + 307t^6 \end{aligned} \tag{6.4}$$

The coefficients of these polynomials play for the crossing loop case the same role as that played by refined alternating sign matrix numbers for the non-crossing one [16]. Note that they share with them the property that at $t = 0$ one recovers the total number of the preceeding rank (with $t = 1$). We have also been able to construct counterparts for the doubly-refined alternating sign matrix numbers [17], by specializing our model to $z_1 = (t-1)/(t+1)$ and $z_{n+1} = (u-1)/(u+1)$ while all other z_i are zero, in which case we still find that all entries of a suitably normalized Ψ_n are polynomials of t, u with non-negative integer coefficients. All these integers await some combinatorial interpretation.

Such an interpretation should be provided by a vertex-type model with fixed boundary conditions on the square grid of size $n \times n$, generalizing the six-vertex model with domain wall boundary conditions of the non-crossing case. The obvious candidates are the $OSP(p|2m)$ vertex models of [7,8], with $p - 2m = 1$, but the main problem is that there are many such models, whose row-to-row transfer matrix acts on a Hilbert space of dimension $(p+2m)^{2n}$ for a system of size n , and that in order to cover at least the $(2n-1)!!$ dimensions of the Hilbert space of crossing link patterns, both p and m should be at least of the order of the size n . This is the worst scenario that could occur, as the number of spin degrees of freedom on each edge of the lattice must itself grow with the size of the

system. The multiplicity of formulations of the vertex model of the non-crossing case as: alternating sign matrices, fully-packed loops, osculating paths, etc leaves much room for imagining a generalization adapted to the crossing loop case, but this is yet to be found.

Acknowledgments

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Appendix A. Entries of Ψ_2

The entries of Ψ_2 read:

$$\Psi \begin{array}{c} \text{1} \\ \diagup \quad \diagdown \\ \text{2} \quad \text{3} \end{array} \begin{array}{c} \text{4} \\ \diagdown \quad \diagup \\ \text{3} \quad \text{2} \end{array} = (1 + z_1 - z_2)(1 + z_2 - z_3)(1 + z_3 - z_4)(1 + z_4 - z_1)$$

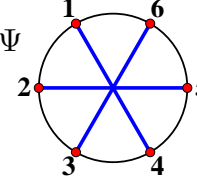
$$\Psi \begin{array}{c} \text{1} \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{3} \end{array} = (1 + z_2 - z_3)(1 + z_4 - z_1) \times ((1 + z_3 - z_1)(2 - z_4 + z_1) + (1 + z_1 - z_2)(1 + z_1 - z_3))$$

$$\Psi \begin{array}{c} \text{1} \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{3} \end{array} = (1 + z_1 - z_2)(1 + z_3 - z_4) \times ((1 + z_4 - z_2)(2 - z_1 + z_2) + (1 + z_2 - z_3)(1 + z_2 - z_4))$$

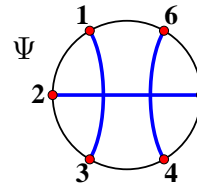
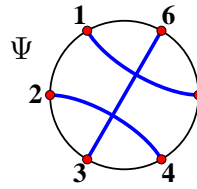
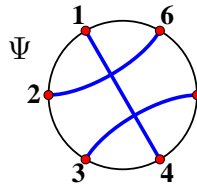
Appendix B. Entries of Ψ_3 from the action of the Θ_i

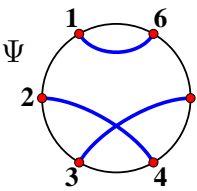
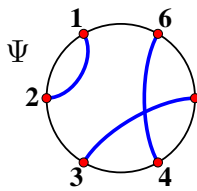
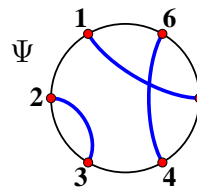
We show here how to obtain each entry of Ψ_3 by repeated use of Eq. (3.11).

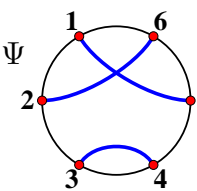
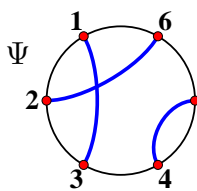
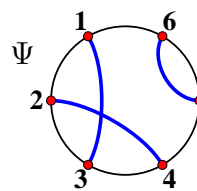
The coefficient Ψ_{π_0} is given by Eq. (3.19). It reads

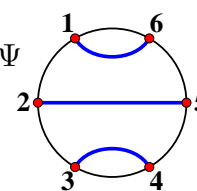
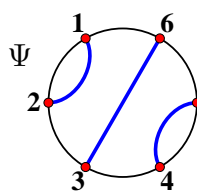
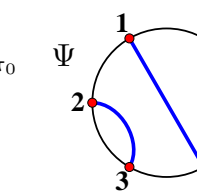
$$\Psi_{\pi_0} = (1 + z_1 - z_2)(1 + z_1 - z_3)(1 + z_2 - z_3)(1 + z_2 - z_4) \times (1 + z_3 - z_4)(1 + z_3 - z_5)(1 + z_4 - z_5)(1 + z_4 - z_6) \times (1 + z_5 - z_1)(1 + z_6 - z_1)(1 + z_6 - z_2)(1 + z_5 - z_6)$$


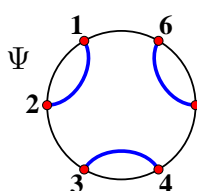
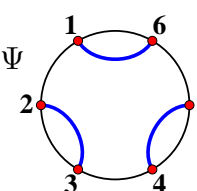
It has total degree 12. The other coefficients are given by


 $\Psi = \Theta_3 \Psi_{\pi_0}$

 $\Psi = \Theta_1 \Psi_{\pi_0}$

 $\Psi = \Theta_2 \Psi_{\pi_0}$


 $\Psi = \Theta_1 \Theta_2 \Psi_{\pi_0}$

 $\Psi = \Theta_2 \Theta_3 \Psi_{\pi_0}$

 $\Psi = \Theta_1 \Theta_3 \Psi_{\pi_0}$


 $\Psi = \Theta_2 \Theta_1 \Psi_{\pi_0}$

 $\Psi = \Theta_5 \Theta_3 \Psi_{\pi_0}$

 $\Psi = \Theta_4 \Theta_3 \Psi_{\pi_0}$


 $\Psi = \Theta_2 \Theta_1 \Theta_2 \Psi_{\pi_0}$

 $\Psi = \Theta_3 \Theta_2 \Theta_3 \Psi_{\pi_0}$

 $\Psi = \Theta_1 \Theta_4 \Theta_3 \Psi_{\pi_0}$


 $\Psi = \Theta_6 \Theta_2 \Theta_1 \Psi_{\pi_0}$

 $\Psi = \Theta_2 \Theta_5 \Theta_3 \Psi_{\pi_0}$

Unfortunately lack of space does not permit us to produce them explicitly here.

Note that we could have restricted ourselves to the computation of the Ψ_π , taking for π one representative in each orbit under rotation, and extended the result to the remainder of the orbit by use of the cyclic covariance property (2.9).

Although we have been able to bypass (3.16) by only using (3.11), we may have applied it to determine the last two entries above

$$\Psi \begin{array}{c} \text{1} \\ \text{2} \quad \text{3} \\ \text{4} \quad \text{5} \\ \text{6} \end{array} = (\Delta_6 \Theta_2 \Theta_1 \Theta_2 - \Theta_2 \Theta_1 - \Theta_3 - 1) \Psi_{\pi_0}$$

$$\Psi \begin{array}{c} \text{1} \\ \text{2} \quad \text{3} \\ \text{4} \quad \text{5} \\ \text{6} \end{array} = (\Delta_3 \Theta_2 \Theta_1 \Theta_2 - \Theta_1 \Theta_2 - \Theta_3 - 1) \Psi_{\pi_0}$$

As already mentioned, both Δ_6 and Δ_3 could in principle increase the degree by one unit, but they both act on the entry $\Theta_2 \Theta_1 \Theta_2 \Psi_{\pi_0}$, whose maximal degree (12) contribution reads

$$\Psi^{max} \begin{array}{c} \text{1} \\ \text{2} \quad \text{3} \\ \text{4} \quad \text{5} \\ \text{6} \end{array} = (z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_1 - z_5)(z_2 - z_3)(z_2 - z_4) \\ \times (z_2 - z_6)(z_3 - z_5)(z_3 - z_6)(z_4 - z_5)(z_4 - z_6)(z_5 - z_6)$$

according to Lemma 2, Eq. (3.27). This is clearly invariant both under $z_6 \leftrightarrow z_1$ and $z_3 \leftrightarrow z_4$, and therefore the degree is preserved by both operators.

Appendix C. Bidegree of the affine homogeneous variety V_π for $n = 3, 4$

The specialized entries Ψ_π , conjectured to be bidegrees of the V_π , are listed below in decreasing number of crossings of the corresponding link pattern.

The 6 bidegrees at $n = 3$:

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = A^3 B^3$$

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = A^2 B^2 (A^2 + AB + B^2)$$

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = A^2 B^2 (A^2 + AB + B^2)$$

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = AB(A^4 + 2A^3B + 4A^2B^2 + 4AB^3 + 2B^4)$$

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = AB(2A^4 + 4A^3B + 4A^2B^2 + 2AB^3 + B^4)$$

$$d \begin{array}{c} \text{1} \quad \text{6} \\ \text{2} \quad \text{5} \\ \text{3} \quad \text{4} \end{array} = A^6 + 3A^5B + 7A^4B^2 + 9A^3B^3 + 7A^2B^4 + 3AB^5 + B^6$$

The 24 bidegrees for $n = 4$:

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^6 B^6$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^5 B^5 (A^2 + AB + B^2)$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^4 B^4 (A^2 + AB + B^2)^2$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^4 B^4 (A^4 + 2A^3 B + 4A^2 B^2 + 4AB^3 + 2B^4)$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^4 B^4 (2A^4 + 4A^3 B + 4A^2 B^2 + 2AB^3 + B^4)$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^3 B^3 (A^6 + 3A^5 B + 7A^4 B^2 + 9A^3 B^3 + 7A^2 B^4 + 3AB^5 + B^6)$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^3 B^3 (A^6 + 3A^5 B + 9A^4 B^2 + 17A^3 B^3 + 21A^2 B^4 + 15AB^5 + 5B^6)$$

$$d \begin{array}{c} \text{1} \quad \text{8} \\ \text{2} \quad \text{7} \\ \text{3} \quad \text{6} \\ \text{4} \quad \text{5} \end{array} = A^3 B^3 (5A^6 + 15A^5 B + 21A^4 B^2 + 17A^3 B^3 + 9A^2 B^4 + 3AB^5 + B^6)$$

$$\begin{aligned}
d \begin{array}{c} \text{Diagram 1} \end{array} &= d \begin{array}{c} \text{Diagram 2} \end{array} = A^3 B^3 (A^2 + AB + B^2) \\
&\quad (2A^4 + 4A^3 B + 5A^2 B^2 + 4AB^3 + 2B^4) \\
d \begin{array}{c} \text{Diagram 3} \end{array} &= d \begin{array}{c} \text{Diagram 4} \end{array} = A^2 B^2 (A^8 + 4A^7 B + 13A^6 B^2 + 28A^5 B^3 + 42A^4 B^4 \\
&\quad + 42A^3 B^5 + 28A^2 B^6 + 12AB^7 + 3B^8) \\
d \begin{array}{c} \text{Diagram 5} \end{array} &= d \begin{array}{c} \text{Diagram 6} \end{array} = A^2 B^2 (3A^8 + 12A^7 B + 28A^6 B^2 + 42A^5 B^3 + 42A^4 B^4 \\
&\quad + 28A^3 B^5 + 13A^2 B^6 + 4AB^7 + B^8) \\
d \begin{array}{c} \text{Diagram 7} \end{array} &= A^2 B^2 (3A^8 + 12A^7 B + 27A^6 B^2 + 40A^5 B^3 + 45A^4 B^4 + 40A^3 B^5 \\
&\quad + 27A^2 B^6 + 12AB^7 + 3B^8) \\
d \begin{array}{c} \text{Diagram 8} \end{array} &= AB(A^{10} + 5A^9 B + 19A^8 B^2 + 47A^7 B^3 + 81A^6 B^4 + 101A^5 B^5 + 97A^4 B^6 \\
&\quad + 73A^3 B^7 + 41A^2 B^8 + 15AB^9 + 3B^{10}) \\
d \begin{array}{c} \text{Diagram 9} \end{array} &= AB(3A^{10} + 15A^9 B + 41A^8 B^2 + 73A^7 B^3 + 97A^6 B^4 + 101A^5 B^5 \\
&\quad + 81A^4 B^6 + 47A^3 B^7 + 19A^2 B^8 + 5AB^9 + B^{10}) \\
d \begin{array}{c} \text{Diagram 10} \end{array} &= AB(2A^{10} + 10A^9 B + 34A^8 B^2 + 82A^7 B^3 + 141A^6 B^4 + 169A^5 B^5 \\
&\quad + 141A^4 B^6 + 82A^3 B^7 + 34A^2 B^8 + 10AB^9 + 2B^{10}) \\
d \begin{array}{c} \text{Diagram 11} \end{array} &= A^{12} + 6A^{11} B + 25A^{10} B^2 + 70A^9 B^3 + 141A^8 B^4 + 210A^7 B^5 + 239A^6 B^6 \\
&\quad + 210A^5 B^7 + 141A^4 B^8 + 70A^3 B^9 + 25A^2 B^{10} + 6AB^{11} + B^{12}
\end{aligned}$$

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